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BPS states and algebras from quivers

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Abstract: We discuss several aspects of D-brane moduli spaces and BPS spectra near orbifold points. We give a procedure to determine the decay products on a line of marginal stability, and we define the algebra of BPS states in terms of quivers. These issues are illustrated in detail in the case of type-IIA theory on $\mathbb{C}^2/\mathbb{Z}_N$. We also show that many of these results can be extended to arbitrary points in the compactification moduli space using Π-stability.

Keywords: Superstrings and Heterotic Strings, D-branes.

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1. Introduction

D-branes provide a window to non-perturbative aspects of string theory. For compactifications with enough supersymmetry, the spectrum is well understood. On the other hand, obtaining the full non-perturbative spectrum of type II on Calabi-Yau varieties is an open challenge. The cohomology of the variety, or more generally

K-theory [1], provides a relevant piece of information, since it determines the allowed charges of D-branes, or in other words, the charge lattice, but finding which sites on that lattice are actually occupied and with which degeneracy is a harder problem.

Recently, a framework for the study of classical BPS branes at arbitrary points in the compactification moduli space of type-II theory on Calabi-Yau varieties has started to emerge [2, 3, 4]. The natural language in this framework is that of homological algebra and category theory, D-branes being the objects in a category, and the fermionic zero modes of the strings stretching between the branes playing the role of the morphisms in the category. Some of the usual notions of this language, as sub-objects or extension groups, have already found a role in the physics, but there is still a long way to go to develop a complete physical intuition about this new language. In this sense, orbifold points seem specially helpful: many of the problems we are interested in (e.g. existence or not of boundstates with given charges, determination of lines of marginal stability) get reduced to questions in linear algebra, and at these corners of moduli space the abstract concepts of the general setup boil down to simple manipulations of matrices. The mathematics relevant for this family of problems goes by the name of quiver theory [5]. One of the purposes of the present paper is to show how the abstract notions of the general framework are easily visualized when we work near orbifold points.

Now that the tools to obtain the (classical) spectrum of BPS branes on Calabi-Yau compactifications are starting to be unraveled, it seems appropriate to rethink what are the likely lessons to be learned from the knowledge of this spectrum. A direction we feel is begging for further exploration is based on the following fact: BPS states form an algebra [6, 7]. What is the significance of this algebra? An analogy that we have in mind are the chiral rings of $\mathcal{N} = 2$ conformal theories [8]. In those theories one focuses on a particular set of states, the chiral primaries, that form a closed algebraic structure, and encode a good deal of information about the full theory. One might wonder to what extent the algebra of BPS states can play a similar role.[1]

Our final goal is to find the quantum spectrum of D-branes, and the classical spectrum is just a first step in that direction. For true bound states, finding the degeneracy involves the quantization of the collective coordinates, and the BPS spectrum turns out to be given by the cohomology of the classical D-brane moduli space.[2] So one issue we must address is how to obtain D-brane moduli spaces at arbitrary points in compactification moduli space. In practice, as we will discuss extensively, constructing moduli spaces involves a notion of stability for the con-

[1] In [7], it is shown that the algebra of BPS states of a two-dimensional $\mathcal{N} = 2$ model is deeply related to the chiral ring.

[2] For boundstates at threshold one has to be more careful and restrict for example to the small diagonal [6]. A full quantum mechanical treatment would involve a study along the lines of [9].

figurations we are considering. In this paper we discuss in detail how to construct D-brane moduli spaces starting from the stability conditions. A key point here is the notion of S-equivalence, which gives the right framework to treat objects that are not stable, but only semistable. We also discuss these issues for the Π-stability condition, which was proposed in [2] as the criterion for stability valid at arbitrary points in moduli space. In particular, we show that S-equivalence can be also introduced in the context of Π-stability.

Another aspect of Π-stability is that it predicts the lines of marginal stability in the compactification moduli space. We will extend this proposal by giving a procedure to obtain also the "decay products" whenever the BPS spectrum jumps by crossing a line of marginal stability.

After this study of the moduli spaces of D-branes, we define the algebra of BPS states near an orbifold point using quiver theory and representations of quivers. To do this we adapt the correspondence conjecture of Harvey and Moore [6] to the orbifold point. In this regime, the correspondence conjecture involves only linear algebra and the computations become considerably easier than in the large volume limit.

All these notions are illustrated in detail in the case of type-IIA theory on $\mathbb{C}^2/\mathbb{Z}_N$. This is a well-known example which has been studied from many points of view, so it is a very good testing ground for the above ideas. First we analyze the BPS spectrum of D2-D0 bound states by looking at stable representations of the quiver, and we reproduce the physical expectations coming from M-theory [10]. We also show that, although, as expected, there are no jumps in the spectrum, there is a rich structure of lines of marginal stability. Next we consider the BPS algebra associated to these states, formulated in terms of representations of quivers. Because of heterotic/type-II duality and the results of [6], this algebra should be the subalgebra of positive roots of the affine $\mathrm{SU}(N)$. We show that this is the case in some non-trivial examples by direct computation from the correspondence conjecture.

The organization of the paper is as follows. In section 2 we review some aspects of the construction of D-brane moduli spaces, emphasizing the notion of stability. We extend this well-known construction to arbitrary points in moduli space, using the recent proposal of Π-stability [2]. As a welcome spin-off, we extend this proposal, providing now a way to obtain the "decay products" along a line of discontinuity of the spectrum. In section 3 we revisit the definition of algebras of BPS states [6], reformulating it near orbifold points in quiver language. We also comment on the possible non-renormalization of BPS algebras with 16 supercharges. In section 4 we illustrate many of these ideas in a simple example: we deduce the spectrum and algebra of BPS states of type IIA on $\mathbb{C}^2/\mathbb{Z}_N$, and reproduce what is known on physical grounds. Finally, we state our conclusions and prospects for future work.

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2. D-brane moduli spaces

2.1 The general picture

Since our discussion will be happily jumping from one point to another in compactification moduli space, before descending into the details, we would like to provide an eagle's view of the general landscape.

We want to describe the spectrum of BPS D-branes for type-II string theory compactified on a Calabi-Yau variety (see [11] for background; some recent references are [12]). A basic feature is that we have two types of D-branes [13, 14] : A-type, wrapping special lagrangian submanifolds in the large volume, and B-type, wrapping holomorphic cycles. Since much more is known about holomorphic cycles, we focus on B-type branes, although one expects that eventually, by mirror symmetry, all the results will have a translation to the A-type branes. Some results for A-type branes have appeared in [15].

Locally, the compactification moduli space splits into complex structure moduli space times Kähler moduli space. We will fix the complex structure, and consider the spectrum of BPS branes at different points in Kähler moduli space. In particular, two kinds of points are under specially good control: the large volume limit and orbifold points. In the large volume, D-branes carry a holomorphic vector bundle, so its classification is related to that of holomorphic bundles, with extra constraints. Near orbifold points, D-branes are described by representations of quivers [16]. In some cases one can construct a map between these two classes of objects, and this was first shown in [17].

Beyond knowing how to describe D-branes themselves, we need to introduce a notion of *stability*. This idea of stability serves a double purpose. First, it allows us to discuss the lines of marginal stability, where the BPS spectrum can jump. Second, it provides a practical way to construct D-brane moduli spaces, bypassing the need of solving all the equations that define the vacua. In the large volume, this notion of stability is known as μ-stability, and it depends on the Chern classes of the bundle. Near orbifold points the relevant notion is θ-stability.

Once we leave the safe havens of the large volume or the orbifold points, we are in uncharted waters. It is time to admit that we do not know what kind of beasts are D-branes in the middle of moduli space! However, the understanding of the two points in moduli space mentioned before gives some clues. The minimal answer seems to be that D-branes are objects in a category defined by a holomorphic constraint [18, 2, 4].[3] Recall that a category is just a set of objects and maps between them (see [21] for a very clear introduction to category theory). D-branes are to be thought as the objects of a category, and the fermionic zero modes of the strings

[3]The role of categories for heterotic string compactifications has been discussed in [19]. Related ideas are developed in [20].

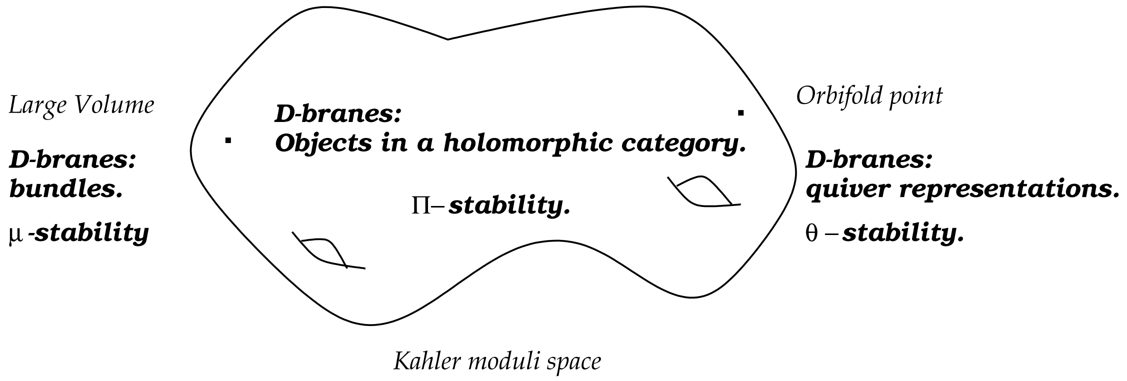


Figure 1: D-branes and stability at generic points in Kähler moduli space and in particular limits.

stretching between them as the maps of the category. This might sound very vague, and certainly it is as it stands, but as we will see, further physical input will help us to narrow down the kind of category we should be considering.

Furthermore, we need to extend the notion of stability to the whole of Kähler moduli space. In this sense, a proposal was presented in [2], where it was dubbed Π-stability, since it involves the periods $\Pi(u)$ of the Calabi-Yau variety. It was proven in [2] that Π-stability reduces to μ-stability and θ-stability in the corresponding limits. The general picture is displayed in figure 1.

2.2 D-branes and quivers

As shown in [16], the description of D-branes near orbifold points involves supersymmetric gauge theories constructed from quivers. A quiver is simply a graph with a set of vertices or nodes V, and a set of arrows A going from node i to node j. For each arrow a we denote by ia its initial node and by ta its terminal node. Given a quiver, one can obtain a gauge theory by considering a *representation* of the quiver. This quiver gauge theory encodes the information about the classical boundstate spectrum and the lines of marginal stability near orbifold points. In [2, 3] it was pointed out that quiver theory allows to extract that information. Let's recall then the basic definitions.

A representation of a quiver is given by a (complex) vector space for each node and a matrix for each arrow: $\{U_v, v = 1, \ldots, V;\ \phi_a : U_{ia} \to U_{ta}\}$. The dimension vector of a representation is a vector with V components given by the dimensions of the vector spaces: $n = (\dim U_1, \ldots, \dim U_V)$. Notice that a representation of the quiver corresponds to a configuration of an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $G = \prod_{v\in V} U(n_v)$, and with chiral multiplets x_a associated to the different arrows between the nodes. These multiplets are in the bifundamental representation $(n_{ta}, \overline{n}_{ia})$.

Since the configuration space of the $\mathcal{N} = 1$ gauge theory is given, roughly speaking, by the space of representations with a fixed dimension, it is important to understand this space in some detail. We will then give some mathematical definitions which will be very useful.

We will say a representation is *indecomposable* if it cannot be written as direct sum of two representations. It can happen that a representation is indecomposable, but "small perturbations" of it are not. When in a neighborhood of an indecomposable representation all representations are indecomposable, we call it a *stably indecomposable* representation, or *Schur* representation. One can prove that a representation R is Schur if and only if $\operatorname{End} R = \mathbb{C}$. The importance of Schur representations stems from this property, since it implies that D-brane boundstates are given by Schur representations [3]. The dimension vector of a Schur representation is called a *Schur root.*

Given two representations R and S, we can define an homomorphism between representations by a collection of linear maps, one per node, $\phi(v) : R(v) \to S(v)$ such that for all arrows, $\phi(R(a)) = S(\phi(a))$. If the homomorphism $\phi : R \to S$ is injective, we say that R is a *subrepresentation* of S. For any ϕ, $\operatorname{Im}\phi$ is a subrepresentation of S and $\operatorname{Ker}\phi$ is a subrepresentation of R. if the generic representation with dimension vector n has a subrepresentation with dimension vector n', we say that n' is a *subvector* of n. As we will see, the relevance of the subrepresentations of a representation is that as we wander in moduli space, subrepresentations are the candidates to destabilize the original representation, causing the BPS spectrum to jump [2].

2.3 μ-stability and θ-stability

Both at the orbifold and in the large volume, BPS configurations are given by solutions to two kinds of equations, modulo gauge equivalence by the appropriate gauge group G. The first one is a holomorphic constraint: in the large volume it implies that we have to restrict to holomorphic bundles, $F^{2,0} = 0$, and at the orbifold point, one has the F-flatness conditions coming from the superpotential, a holomorphic quantity.

The second kind of equation imposes the BPS condition. Near the orbifold point this is the D-flatness condition, and in the large volume, the equation is $F^{1,1} = \zeta\omega$ or some deformation thereof [22], where ω is the Kähler form. It turns out that both these equations can be reformulated as a (quasi)-topological condition.[4] These reformulations at very different points in moduli space share many common properties, and lead naturally to a generalization of the (quasi)-topological condition for arbitrary points in moduli space.

This reformulation arises as follows. In order to solve the equations, it proves useful to solve first the complex one. For example, in the large volume limit, this corresponds to considering connections that define a holomorphic bundle. Second,

[4]The reason that we say (quasi)-topological is that it depends explicitly on Kähler moduli.

since the space of solutions to the complex equations is invariant under the action of the *complexified* gauge group $G_{\mathbb{C}}$, one can talk about the $G_{\mathbb{C}}$-orbits satisfying the complex equations. The key result is that, under certain conditions, these complex orbits contain a solution of the real equation. This condition is called *stability*.

In the large volume limit, stability refers to the usual Mumford stability condition for holomorphic vector bundles (see, for example, [23, 24], and also [25] for a physical point of view): for a bundle F of rank $r(F)$ on a Kähler manifold, define its degree as

$$\deg(F) = \int c_1(F) \wedge \omega^{d-1}\,, \tag{2.1}$$

and its slope μ as

$$\mu(F) = \frac{\deg(F)}{\mathrm{r}(F)}\,. \tag{2.2}$$

The bundle F is μ-semistable if and only if for every subbundle F' of F we have

$$\mu(F') \leq \mu(F)\,. \tag{2.3}$$

If $\mu(F') < \mu(F)$ we say that F is μ-stable. Note that $\mu(F)$ depends explicitly on Kähler moduli through ω. The relevance of μ-stability is the following: if the bundle is holomorphic, the holomorphic structure is defined by a $G_{\mathbb{C}}$-orbit of gauge connections satisfying $F^{2,0} = 0$. If the bundle is stable, then there is one representative in this orbit which solves the real equation $F^{1,1} = \zeta\omega$.

Near the orbifold point, the world-volume of D-branes is given by quiver gauge theories [16], and D-brane configurations correspond to representations of quivers which satisfy the F-flatness and D-flatness conditions. We write these conditions schematically as:

$$\frac{\partial W(\phi)}{\partial \phi} = 0\,, \qquad \sum [\phi, \phi^\dagger] = \theta\,, \tag{2.4}$$

where $W(\phi)$ is the superpotential depending on the chiral fields ϕ. In the second equation, the right hand side is given by a block-diagonal matrix whose entries are $\theta_v \cdot \mathbf{1}_{n_v \times n_v}$, $v = 1, \ldots, V$. In this case, the appropriate notion of stability is θ-stability [26]. Let $\theta = (\theta_1, \ldots, \theta_V)$ be a vector whose V components are real numbers. A representation of a quiver is called θ-semistable if $\sum_v n_v \theta_v = 0$, and for every subrepresentation with dimension vector n', one has:

$$\sum_i n'_i \theta_i \geq \sum_i n_i \theta_i = 0\,. \tag{2.5}$$

If $\sum_i n'_i \theta_i > 0$ we say that the representation is θ-stable. As in the large volume limit, the procedure to solve the equations (2.4) is to solve first the complex ones, and consider complexified orbits of the solutions. The complexified gauge group is $G_{\mathbb{C}} = \prod_v \mathrm{Gl}(n_v, \mathbb{C})/D$, where we quotient by the diagonal $D = (\lambda \cdot \mathbf{1}_{n_1 \times n_1}, \ldots, \lambda \cdot \mathbf{1}_{n_V \times n_V})$, $\lambda \in \mathbb{C}^*$. A theorem of King [26] guarantees that, if the representation is θ-stable, the $G_{\mathbb{C}}$-orbit will contain a solution to the D-flatness conditions.

The vector θ is closely related to the physical Fayet-Iliopoulos (FI) terms. For $\mathbb{Z}_N$-orbifolds, the relation is given by [2]:

$$\theta = \zeta - \frac{\zeta \cdot n}{e \cdot n} e \,, \tag{2.6}$$

where $e = (1, 1, \ldots, 1)$, and $\zeta = (\zeta_1, \ldots, \zeta_N)$ are the FI terms.[5] Notice that $\sum_v \zeta_v = 0$, therefore $\theta \cdot n = 0$. The condition (2.6) is obtained after taking into account that the D-flatness conditions of (2.4) describe in fact quasi-supersymmetric vacua with a constant, non-zero value for the vacuum energy [27]. Requiring this energy to be minimal in the sector specified by the dimension vector n gives (2.6). When written in terms of the Fayet-Iliopoulos terms, θ-semistability becomes:

$$\frac{\zeta \cdot n'}{e \cdot n'} \geq \frac{\zeta \cdot n}{e \cdot n} \,, \tag{2.7}$$

for any subrepresentation of dimension n'. Notice the formal similarities between μ-stability and θ-stability: in both cases we have an object (a bundle or a quiver representation), and to satisfy stability it has to obey an inequality (eqs. (2.3) or (2.5)) against the list of all its subobjects (subbundles or subrepresentations). In fact, $e \cdot n$ plays the role of $\mathrm{r}(F)$, while $-\zeta \cdot n$ plays the role of $\deg(F)$.

2.4 Π-stability

In view of the previous considerations, we are led to ask if there is a generalization of these stability criterions that is valid everywhere in moduli space, and reduces to μ or θ stability in the corresponding limits. Since we want a criterion exact in α', if it exists at all, it is natural to suspect that it will involve the periods $\Pi(u)$, since they can be exactly determined using mirror symmetry.

To start with, inspired by the large volume and orbifold examples, it was assumed in [2] that D-branes at generic points in moduli space have also sub D-branes (the generalization of sub-bundles and subrepresentations), so we require that there is a notion of subobject of an object in our category. The technical definition is that E' is subobject of E if there exists an injective homomorphism from E' to E. With this assumption, the proposal of Π-stability is the following [2]: at a point u in Kähler moduli space, define the *grading* of a brane E with central charge $Z(E,u)$[6] by

$$\varphi(E,u) = \frac{1}{\pi} \operatorname{Im} \log Z(E,u) \,. \tag{2.8}$$

[5]In this paper, we will always choose $\zeta_{\mathbb{C}} = 0$, so $\zeta = \zeta_{\mathbb{R}}$.

[6]As explained in detail in [2], it is necessary to extend slightly the usual notion of central charge, by keeping track of the integer part of the phase as we wander in moduli space

$$Z(E,u) = \Pi(u) \cdot Q(E) \; e^{2\pi i n(E,u)} \,.$$

We say that E is Π-semistable at u if and only if for all the sub-branes E' of E at u we have

$$\varphi(E) \geq \varphi(E')\,. \tag{2.9}$$

If we have strict inequality, we say that E is Π-stable. It was shown in [2] that this definition reduces to μ-stability in the large volume and to θ-stability at orbifold points. This is displayed in figure 1 above.

Recall that we started with two sets of equations at very different points in moduli space. Some of these equations admitted a translation into a stability condition, and it was in this guise that a generalization was proposed in [2]. This raises the interesting question of what is the equation that corresponds to Π-stability, in the same sense as the D-term equation gives rise to θ-stability at orbifold points or $F^{1,1} = \zeta\omega$ translates into μ-stability near the large volume.

2.5 Moduli spaces and S-equivalence

In the previous subsection we have reviewed how the D-brane moduli spaces arise as solutions to classical equations with gauge invariance. In both cases, the usual way to construct the moduli spaces of solutions is by exploiting the correspondence between the solutions and a suitable category of stable objects (μ-stable bundles in the large volume, θ-stable representations at the orbifold point). Let's now discuss the orbits which are not stable, but only semistable.

The situation for semistable orbits is more complicated, since some of them do not contain any solution to the classical equations, and some of them do. This means that some semistable representations are not really relevant to the solution of the moduli equations. On the other hand, it is highly desirable to describe this moduli space in terms of representations obeying (semi)stability conditions. The solution to this problem is to introduce an equivalence relation in the set of semistable orbits in such a way that, for each equivalence class, there is one and only one solution to the real equation. This relation is called *S-equivalence.* Among the representations which are in the S-equivalence class, there is one which is the actual solution to the moduli equations, and it is called the graded representation. As we will see, S-equivalence also provides a recipe to obtain this graded representation. It is important to stress that S-equivalence, at least in this situation, is an artifact of our description: we have chosen to describe the moduli space of solutions to our equations in terms of holomorphic objects, but in the semistable case some of these objects are spurious. S-equivalence is the proper way to get rid of the extra objects, by identifying them to the honest solution to the original equations. We will explain here in some detail the definition of S-equivalence in the case of representations of quivers [26]. A discussion of S-equivalence in the context of heterotic compactifications can be found in [19, 25].

To give a precise definition of S-equivalence, we have to make some technical definitions. First of all, we define the *Jordan-Hölder filtration* of a θ-semistable representation R as an increasing filtration of θ-semistable representations R_i:

$$0 = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n = R\,, \tag{2.10}$$

satisfying two conditions (see for example [28]): first, $\theta \cdot (\dim R_i) = 0$, where $(\dim R_i)$ is the dimension vector of R_i; second, the quotients $M_i = R_i/R_{i-1}$ are θ-stable (in particular, $M_1 = R_1$ is a stable subobject of R). One can prove that any semistable representation has a Jordan-Hölder filtration. Notice that, if R is θ-stable, then one can take as a Jordan-Hölder filtration just $0 \subset R$. We now define the *graded representation* $\mathrm{gr}(R)$ as:

$$\mathrm{gr}(R) = \oplus_i M_i\,. \tag{2.11}$$

The main point is that, although the Jordan-Hölder filtration of R is not necessarily unique, the graded representation of R is unique (up to isomorphism). If R is θ-stable, its graded representation is R itself. We can now define S-equivalence. Let R, $\tilde{R}$ be θ-semistable representations. We say that R and $\tilde{R}$ are S-equivalent (and we write $R \sim_S \tilde{R}$) if $\mathrm{gr}(R)$ and $\mathrm{gr}(\tilde{R})$ are isomorphic representations. There are two immediate consequences of this: first of all, two strictly stable representations are S-equivalent if and only if they are isomorphic. Second, a representation R is always S-equivalent to $\mathrm{gr}(R)$.

Notice that there are representations which are S-equivalent but are not in the same $G_{\mathbb{C}}$-orbit. The $G_{\mathbb{C}}$-orbit which actually contains the solution to the equations is the one which is a *direct sum* of θ-stable representations [23]. This means that, as we have stressed above, given a θ-semistable representation R, the solution to our equations will be in the $G_{\mathbb{C}}$-orbit of the graded representation $\mathrm{gr}(R)$. We will see that the graded representation associated to a representation has in fact a very clear physical meaning: it corresponds to the "decay products" or "primary constituents" [29] of the BPS state represented by R when we are on a line of marginal stability.

In conclusion, the D-brane moduli space we are looking for can be realized as the space of S-equivalence classes of semistable $G_{\mathbb{C}}$-orbits which solve the complex equations. In the orbifold limit, the space of θ-semistable representations with dimension vector n will be denoted as $\mathcal{M}_\theta(n)$. Since, for a given n, θ is related to ζ by (2.6), we will also denote this space by $\mathcal{M}_\zeta(n)$.

Although the description of the space of solutions in terms of stable objects sounds rather formal, it is of practical use: in order to find the moduli space of solutions, instead of solving the real and the complex equations modulo usual gauge invariance, one can just solve the complex equation modulo complexified gauge transformations, and look for semistable objects up to S-equivalence. We will give examples of this procedure later on.

2.6 S-equivalence and Π-stability

Since our goal is to describe D-brane moduli spaces everywhere in Kähler moduli space, we should try to make sense of S-equivalence for the whole of moduli space. As we have explained, S-equivalence requires a notion of stability, and Π-stability is our only current candidate, so we will try to extend the notion of S-equivalence everywhere in moduli space using Π-stability.

In doing so, we assume explicitly, following [2], that the category of D-branes is abelian.[7] First we will present the notion of Jordan-Hölder filtration for an abelian category. Then we will try to argue that at arbitrary points in moduli space, the set of Π-semistable D-branes with fixed grading φ forms an abelian subcategory of the category of all possible D-branes, so the previous generic definition applies to them. The presence of the integral part of the phase in the grading (see footnote 6) will prevent us from giving a complete argument, but we will present a proof in the case of small difference of phases.

Let $\mathcal{A}$ be an abelian category. We say that an object $E \in \mathcal{A}$ is *simple* if it does not have any proper subobject. An increasing sequence of subobjects $0 \subset E_1 \subset E_2 \subset \cdots E_n = E$ is called a Jordan-Hölder series of E if E_1 and the quotients $M_i = E_i/E_{i-1}$ are all simple. Given a Jordan-Hölder series we define

$$\mathrm{gr}(E) = \oplus_i M_i \,,$$

which is the natural generalization of (2.11) to any abelian category.

An abelian category is *artinian* if every decreasing sequence is finite. After all these definitions, we are ready to present the theorem that we will rely upon

Theorem [32]. Let $\mathcal{A}$ be an abelian category. If an object M in $\mathcal{A}$ has a Jordan-Hölder series, then $\mathrm{gr}\, M$ is unique up to isomorphism. Furthermore, if $\mathcal{A}$ is artinian, then every object in $\mathcal{A}$ has a Jordan-Hölder series.

Before we discuss the generic case, let us recall the situation at the orbifold point. There King [26] argues that the category of all θ- semistable representations for a fixed θ is abelian, artinian and noetherian. Furthermore, the simple objects in this category are the θ-stable objects: consider a θ-stable object; any proper subobject should satisfy $\theta \cdot n' > 0$, but by definition all the objects in that category satisfy $\theta \cdot n' = 0$.

[7] We would like to make a remark concerning our assumption of the category of D-branes being abelian [4]. There is a number of papers in the physics literature advocating that derived categories are the right framework to think about D-branes [18, 19]; in fact, this idea predates the modern understanding of D-branes, and originates in the work of Kontsevich on mirror symmetry [30]. Very roughly, if we think of D-branes as objects in a category, the derived category would correspond to identifying D-brane configurations that differ by brane-antibrane pairs. The bad news is that in general derived categories are not abelian (see, for example, [31].) Therefore, at present, the physical input that we are using to try to pin down the precise category that describes D-branes, seems to translate into not obviously compatible mathematical requirements.

In the large volume, there is a difference: μ-stable bundles can have μ-stable subbundles, the only requirement being that the slope is smaller $\mu(E') < \mu(E)$. So the category to look at is that of μ-semistable bundles with fixed slope μ [24]. This is also the generic category to consider at arbitrary points in moduli space: the category $\mathcal{A}_\varphi(u)$ of Π-semistable objects with a fixed grading φ. By the same argument as before, the simple objects in this category are the Π-stable objects. According to the previous theorem, we should argue that $\mathcal{A}_\varphi(u)$ is artinian to prove the existence of a Jordan-Hölder series, and abelian to prove that its graded sum is unique. In the following we will argue that it is abelian. We do not have a rigorous argument showing that it is artinian, but on physical grounds this seems quite plausible: in the next subsection, we will argue that the graded sum gives the decay products on a line of marginal stability. If our proposal is valid, $A_\varphi(u)$ not being artinian would imply that some objects would decay into an infinite number of subobjects on the line of marginal stability.

To prove that this subcategory is actually abelian, we can apply almost verbatim the proof in [24] for bundles in the large volume. There is a small complication, however; when we have an exact sequence of bundles

$$0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0\,, \tag{2.12}$$

the following relation among their slopes follows:

$$\mu(G) = \frac{\mathrm{r}(E)}{\mathrm{r}(G)}\mu(E) + \frac{\mathrm{r}(F)}{\mathrm{r}(G)}\mu(F)\,. \tag{2.13}$$

There is no such relation for the gradings $\varphi(E), \varphi(G), \varphi(F)$. Although this relation among slopes is used repeatedly in [24], we do not need that much. When the difference of phases is small, the gradings satisfy a convexity condition

$$\varphi(G) = x\varphi(E) + (1-x)\varphi(F) \quad \text{for some } 0 \leq x \leq 1 \tag{2.14}$$

(the relation for the slopes is a particular case of this), and this turns out to be enough for the proof to go through.[8] The detailed proof is presented in appendix A.

2.7 Bound states and lines of marginal stability: a proposal for decays

According to the proposal of [2, 3], BPS bound states at the orbifold point are described by θ-stable representations, where the value of θ is related to the physical FI parameters by (2.6). Since θ-stable representations are Schur, this is in agreement with the idea that bound states of D-branes leave a single U(1) unbroken.[9] In general, however, there will be a whole moduli space of θ-semistable representations

[8] We are grateful to M.R. Douglas for this suggestion.

[9] Notice that a representation can be Schur and nevertheless be unstable for some values of the parameters, and therefore does not correspond to a (quasi)-supersymmetric vacuum.

$\mathcal{M}_\zeta(n)$. In the semiclassical approximation, one has to do supersymmetric quantum mechanics on this moduli space, and the spectrum of BPS states will be given by the L^2-cohomology of the moduli space, as in [6]:

$$\mathcal{H}^{n,\zeta}_{\rm BPS} = H^*_{L^2}(\mathcal{M}_\zeta(n))\,. \tag{2.15}$$

We have recorded the dependence of this Hilbert space on the background FI parameters. We should say that the above equation is still very incomplete. For example, as noted in [6], to understand in detail the multiplicity structure of the BPS states it is necessary to consider representations of the supertranslation algebra. However, (2.15) will be enough for our purposes in this paper.

An important feature of the spectrum of BPS states for a theory with 8 supercharges is that it can present discontinuities as we move in moduli space. This was first observed in two dimensions [33], and plays an important role in the celebrated solution by Seiberg and Witten of $\mathcal{N}=2$ SU(2) SYM [34]. It is well known that a necessary condition for the existence of lines of discontinuity in the spectrum is given by

$$\mathrm{Im}\,\frac{Z_1(u)}{Z_2(u)} = 0\,, \tag{2.16}$$

where $Z_{1,2}(u)$ are central charges at the point u in moduli space. The question of when the spectrum does actually jump is much harder. Π-stability provides an answer: the spectrum will jump when, given a vector charge Q, the cohomology of the moduli space of Π-stable objects with that charge changes when passing through the line of marginal stability. This means that the moduli space of stable objects has changed, due to the fact that objects that were stable on one side of the line of marginal stability become unstable on the other side. Therefore, the lines of marginal stability have a very clear mathematical counterpart: they correspond to the values of the background parameters for which stable objects become semistable. At the lines of marginal stability, the best we can have is a BPS state at threshold. To know if this is the case we should address the full quantum-mechanical problem, as in [9]. In this paper we will only use semiclassical considerations, so we are not going to tackle this issue.

Let us now focus on the situation at the orbifold point, and consider for simplicity the case in which $\mathcal{M}_\theta(n)$ is zero-dimensional and consists of a θ-stable representation R. In general, in the parameter space of θ's there are lines of marginal stability where R becomes just θ-semistable. In this case, there will be a non-trivial S-equivalence class of representations, including (at least) R itself and its graded representation: $[R]_S = \{R, \mathrm{gr}(R), \ldots\}$. As discussed in [25] for the large volume, as we move away from the line of marginal stability, and depending on the region we move to, some of the representations in $[R]_S$ will become stable, while the others will become unstable.

There are two possible outcomes of this process:[10]

(1) It can happen that, as we move from the line, there is a θ-stable representation with the same charge. This means that the corresponding BPS state exists on both sides of the line of marginal stability. This will be in fact the case for the examples studied in this paper.

(2) If there is no θ-stable representation with the given charge, we clearly have a jump in the BPS spectrum. This is interpreted sometimes (see for example [35]) by saying that the BPS state has decayed through the line. In this case, we propose that the decay products are given precisely by the graded representation:
$$R \longrightarrow \mathrm{gr}(R) = \oplus_i M_i \,. \tag{2.17}$$

The jump in the spectrum can be interpreted by saying, as in [2], that the object R has been destabilized by a subobject R_1 (notice that R_1 comes from the Jordan-Hölder filtration (2.10) and therefore it is always a subobject of R). Moreover, the graded representation gives the full content of the decay products, and not only a destabilizing subobject. This proposal is very natural, since the graded representation gives the decomposition of the representation into the "minimal" objects, i.e. into the stable pieces that constitute the semistable object. This decomposition is in fact consistent with the central charge criterion, and as we will see later in some examples, (2.17) implies that
$$\|Z(R)\| = \sum_i \|Z(M_i)\| \,. \tag{2.18}$$

The physical mechanism behind the "decay" process has been explained more precisely in [29]. In their language, the products of the decay are in fact "primary constituents" of the BPS state which are no longer bound. Our proposal says which of the possible primary constituents of the object are actually becoming unbound at the line of marginal stability.

The same discussion applies to the large volume limit, where the BPS states are associated to bundles. Again, semistable bundles on the line of marginal stability are S-equivalent to graded bundles. Since we also have Jordan-Hölder filtrations and graded objects in the context of Π-stability, the proposal extends in fact to the whole of the moduli space.

In the discussion above we have considered for simplicity moduli spaces involving one single representation. The argument extends easily to the case in which one has

[10] While this paper was being typed, the paper [29] appeared, where these two possible outcomes were also discussed from a dynamical point of view.

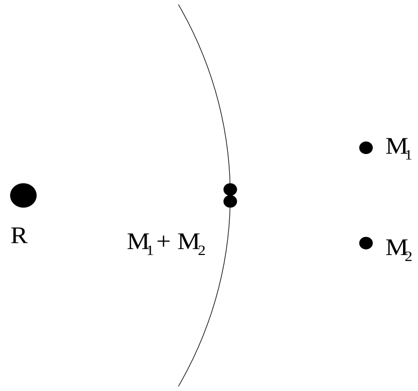


Figure 2: Before arriving to the line of marginal stability, R is stable. On the line it becomes semistable, and S-equivalent to $M_1 \oplus M_2$.

a finite number of representations. When the moduli space has positive dimension and the BPS states are given by the L^2-cohomology, the situation is much more subtle. Probably, there are extra quantum numbers associated to some group action in the Cohomology (like in [36, 37]) that have to be taken into account more carefully.

3. BPS algebras from quivers

3.1 Definition of the BPS algebra

It was shown in [6] that BPS states form an algebra which captures some important non-perturbative aspects. In the large volume limit of type-IIA compactifications, the BPS states associated to wrapped D-branes are described in terms of bundles (or, more generally, of coherent sheaves). The proposal of [6] for the computation of the BPS algebra in this regime is based on the so called "correspondence conjecture". The correspondence conjecture is in general difficult to test, since it involves moduli spaces associated to short exact sequences of bundles. However, it turns out that this conjecture can be adapted readily to the orbifold point, and more generally, to any abelian category with a notion of stability. We will then adapt the construction of [6] to define the algebra of BPS states in terms of representations of quivers. The big advantage of the orbifold point is that computations can be done using linear algebra, and very little geometry.

Consider a quiver, and two dimension vectors n_1 and n_2, as well as their sum $n_3 = n_1 + n_2$. Consider the moduli spaces of semistable representations of this quiver for the different dimensions, $\mathcal{M}_\zeta(n_i)$. We will assume that ζ is chosen in such a way that all the points in these spaces are in fact isomorphism classes of stable representations, and that they are non empty. We now define the correspondence variety $\mathcal{C}(n_1, n_2; n_3)$ as follows:

$$\mathcal{C}(n_1, n_2; n_3) = \Big\{(R_1, R_3, R_2) \in \prod \mathcal{M}_\zeta(n_i) : 0 \longrightarrow R_1 \longrightarrow R_3 \longrightarrow R_2 \longrightarrow 0\Big\} . \quad (3.1)$$

The correspondence variety is then given by the set of isomorphism classes of triples that fit into an exact sequence.

Remarks:

1. As it stands, this definition is incomplete, since we are considering only stable objects. One should also include S-equivalence classes of semistable objects, by extending the above definition in the obvious way. However, one should check that the definition makes sense, i.e. that if $R_{1,2,3}$ fit into a short exact sequence, and $R'_i \sim_S R_i$, then $R'_{1,2,3}$ fit too. We suspect that this is the case, but we do not have a proof. Notice that the same caveat applies to the definition in [6] in terms of bundles.

2. It is easy to prove that, if R_3 is stable, then one can have the short exact sequence $0 \to R_1 \to R_3 \to R_2 \to 0$ or $0 \to R_2 \to R_3 \to R_1 \to 0$, but not both. To see this, notice that

$$(e \cdot n_1) \left\{ \frac{\zeta \cdot n_3}{e \cdot n_3} - \frac{\zeta \cdot n_1}{e \cdot n_1} \right\} + (e \cdot n_2) \left\{ \frac{\zeta \cdot n_3}{e \cdot n_3} - \frac{\zeta \cdot n_2}{e \cdot n_2} \right\} = 0 \, . \quad (3.2)$$

   This is the orbifold analog of the equality (2.13) for the slopes. Therefore, if one has the short exact sequence $0 \to R_1 \to R_3 \to R_2 \to 0$, and R_3 is stable, then R_2 cannot be a subrepresentation of R_3, since this would violate stability.

We are now ready to define the "correspondence product". Recall that BPS states are associated to L^2-cohomology classes of $\mathcal{M}_\zeta(n_i)$. Let $\omega_i \in H^*_{L^2}(\mathcal{M}_\zeta(n_1))$, $\eta_j \in H^*_{L^2}(\mathcal{M}_\zeta(n_2))$, $\psi_k \in H^*_{L^2}(\mathcal{M}_\zeta(n_3))$ be a basis of cohomology classes in the Hilbert spaces. We define the BPS product as in [6]:

$$\omega_i \otimes \eta_j = \sum_k \mathcal{N}^k_{ij} \psi_k \, , \quad (3.3)$$

where the coefficients $\mathcal{N}^k_{ij}$ are given by:

$$\mathcal{N}^k_{ij} = \int_{\mathcal{C}(n_1, n_2; n_3)} \psi^*_k \wedge \omega_i \wedge \eta_j \, . \quad (3.4)$$

In the above integral, the differential forms on the moduli spaces $\mathcal{M}_\zeta(n_i)$ are pull-backed to the correspondence variety in the natural way. The above product defines an algebra structure on the space of BPS states, as in [6]. Given this product we define a bracket as follows:

$$[\omega_i, \eta_j] = \omega_i \otimes \eta_j - \eta_j \otimes \omega_i = \sum_j c^k_{ij} \psi_k \, , \quad (3.5)$$

where the structure constants are given by:

$$c_{ij}^k = \mathcal{N}_{ij}^k - \mathcal{N}_{ji}^k \,. \tag{3.6}$$

Remarks:

1. When the correspondence variety has zero dimension (and this will be the case in some important examples) the coefficients $\mathcal{N}_{ij}^k$ are just integers that count the number of short exact sequences. In that case the algebra that we have defined is essentially given by the Ringel-Hall algebra associated to the quiver diagram (see [38, 40] for a recent discussion). When the moduli space has positive dimension, the algebra of BPS states, as defined above, should be closely related to the algebras defined in this context by Nakajima [41].

2. Our definition is a little bit different from the one given in [6] in terms of bundles, since we make a distinction between the product (3.3) and the bracket (3.5). However, due to our remark 2 above, in many cases N_{ij}^k and $\mathcal{N}_{ji}^k$ cannot be both non zero, so the product (3.3) and the bracket agree up to a sign (a similar situation arises in Ringel-Hall algebras, see [39]).

3. The BPS algebra has a direct physical interpretation: if $c_{ij}^k \neq 0$, then the BPS states associated to ω_i and η_j can form the boundstate ψ_k. Therefore, the BPS algebra carries information about the structure of bound states of the theory. Notice that, if $c_{ij}^k \neq 0$, then one has in particular that $\mathrm{Ext}(R_1, R_2) \oplus \mathrm{Ext}(R_2, R_1) \neq 0$, which is a necessary condition to have a boundstate pointed out in [2]. However, the criterion based on the BPS algebra structure is more precise, since one can have a non-trivial extension without having a boundstate (for example, if the extension is unstable).

4. The values of $\mathcal{N}_{ij}^k$ and c_{ij}^k depend in general on the values of the FI parameters ζ. In the examples the we will consider later on, this dependence is rather mild, and will be given by a change of sign in some of the structure constants when we pass through the lines of marginal stability.

5. The algebra is computed in the approximation where BPS states are identified with cohomology classes in some moduli space. In this approximation, the computation is purely cohomological. In particular, there is no trace of the string coupling constant g_s^{II}. More generally, one can ask what is the dependence of the BPS algebras on the coupling constant. We would like to add some comments on this issue.

3.2 Renormalization of the algebra of BPS states with 16 supercharges

In [6], Harvey and Moore raise the question whether BPS algebras with 16 supercharges do not get renormalized and present some evidence for BPS states annihilated by the same supersymmetry generator, in the case of type IIA on K3/ Het on $\mathbb{T}^4$. This is easier to analyze from the heterotic side.

In toroidal compactifications of the heterotic string, the perturbative BPS states are the Dabholkar-Harvey states [42]. The rightmoving (supersymmetric) sector is in the ground state, whereas the leftmoving (bosonic) sector is arbitrary, only constrained by level matching. Because the rightmoving sector is identical to those of massless states, it is conceivable that the world-sheet non-renormalization arguments given for massless scattering amplitudes extend for BPS states annihilated by the same supercharges, or maybe even arbitrary BPS states [42]. Indeed, note that the 2-point function of these states is exact. More to the point, since their algebra is extracted from the 3-point functions [6], if the world-sheet non-renormalization theorems generalize, that would mean that the algebra of perturbative BPS states for Het on $\mathbb{T}^d$ would be given by the tree level computation, at least perturbatively.

World-sheet arguments for the perturbative non renormalization of massless n-point amplitudes, $0 \leq n \leq 3$, have appeared in a number of papers, starting with [43]. These first arguments overlooked subtleties that arise when integrating out the fermionic moduli, since this causes the supersymmetry current to develop unphysical poles [44]. The residue at these poles can be written as a contribution coming from the boundary of moduli space, when the world-sheet Riemann surface degenerates into two Riemann surfaces connected by a long thin tube, creating an ambiguity in the definition of the scattering amplitude. Those ambiguities were analyzed extensively in [45], whose results are consistent with the claim that these amplitudes are perturbatively not renormalized. It would be interesting to check in detail whether the analysis of [45] carries on to arbitrary BPS states.

Non-perturbatively, we need an space-time approach. For massless states, there is an argument due to Dine and Seiberg [46], showing that the $0 \leq n \leq 3$ massless scattering amplitudes with 16 supercharges are *exact*. However, since we do not have a space-time formalism dealing with arbitrary BPS states, we will just be able to comment on the more obvious sources of non-perturbative effects.

Instanton corrections could come from wrapped euclidean heterotic 5-branes but they cannot contribute until we compactify on $\mathbb{T}^6$. On the other hand, by Shenker's argument [47], we expect $e^{-1/g}$ effects in every string theory, including heterotic theory (see [48] for such effects for heterotic strings). A way to rule out such corrections to the tree level result would be to argue that for heterotic on $\mathbb{T}^6$, the BPS algebra is holomorphic in the complexified heterotic coupling constant τ, since $e^{-1/g}$ is not holomorphic in τ. If so, by decompactifying dimensions of $\mathbb{T}^6$, one could argue that these corrections are not present for higher-dimensional toroidal compactifications either.

In short, there is some evidence, at least for mutually local BPS states, that the algebra of BPS states with 16 supercharges is independent of the coupling constant. From the heterotic side, this seems plausible at the perturbative level. However, currently we lack a framework to study this issue non perturbatively.

4. BPS spectrum and algebra for type IIA on $\mathbb{C}^2/\mathbb{Z}_N$

Compactifications of type-IIA theory on $\mathbb{C}^2/\mathbb{Z}_N$/ALE space are considerably simpler than on a Calabi-Yau threefold, but they are a very good example to illustrate our proposals in previous sections. First we present the physical results (which can be obtained using various dualities) and then we analyze to what extent we can derive them from quiver theory.

4.1 Physical results

Consider IIA on $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of SU(2). It has associated with it a simply laced Dynkin diagram $\mathcal{G}$, an affine algebra $\hat{G}$ and a Lie algebra G. At the orbifold point there are 2-cycles shrunk to zero volume; their intersection matrix is the Cartan matrix of the algebra G. Recall that we can have a B-field at the shrunk cycles [49]. If the B-field is turned on, the conformal field theory is well behaved, whereas if the B-field is zero, the conformal field theory breaks down and we have enhanced gauge symmetry, as required by duality with heterotic string on $\mathbb{T}^4$. We will mostly focus on $\Gamma = \mathbb{Z}_N$, which gives an affine $\hat{A}_{N-1}$ algebra.

The possible BPS D-branes are D2-branes wrapping some set of 2-cycles, possibly carrying in addition the charge of n D0-branes. The mass formula for a brane wrapped about the cycle α carrying n units of D0 charge is

$$m^2 = \frac{1}{g_s^2}\Big((B_\alpha)^2 + (J_\alpha)^2 + n^2\Big)\,, \tag{4.1}$$

where B_α, J_α denote the integral of B and J, respectively, on the α-cycle.[11] The mass formula just tells us the mass that a BPS state present at a site of the charge lattice would have. We want to determine which sites of the charge lattice are indeed occupied by single particle BPS states, and what is the degeneracy of states at each site. In the next section we will see that we can answer these questions at least semiclassically using the methods of [2, 3].

The charge lattice can be identified with the positive root lattice of the affine Lie algebra $\hat{G}$. If we choose a basis of simple roots $\alpha_0, \alpha_1, \ldots, \alpha_n$, where n is the rank of G, this lattice is

$$\Gamma_+ = \Big\{k_1\alpha_1 + \cdots + k_n\alpha_n + k_0\alpha_0 \,|\, k_i \geq 0\Big\}\,. \tag{4.2}$$

The simple roots $\alpha_1, \ldots, \alpha_n$ correspond to the exceptional divisors that one obtains after resolving the singularity. The null root is defined as $\delta = \sum_i m_i\alpha_i$, where m_i are the usual labels of the affine Dynkin diagram. For $\widehat{A}_{N-1}$, one has $\delta = \alpha_1 + \cdots + \alpha_0$, and the B-field at the perturbative orbifold is given by the periods $B_{\alpha_i} = 1/N$ for the simple roots [49, 10]. We will always consider the orbifold with the B-field turned on. Also notice that the integrals of J on the cycles α_i are the FI terms: $J_{\alpha_i} = \zeta_i$.

[11] As noted above, we are choosing $\zeta_{\mathbb{C}} = 0$.

We can write the mass m_α for a BPS state with charge α as $m_\alpha = \|Z_\alpha\|/g_s$, where Z_α is the central charge given by:

$$Z_\alpha = \zeta \cdot \alpha + i\frac{e \cdot \alpha}{N}\,. \tag{4.3}$$

In this equation, $e = (1, 1, \ldots, 1)$ as in (2.6), and we have denoted $\zeta \cdot \alpha = \sum_i k_i \zeta_i$, $e \cdot \alpha = \sum_i k_i$.

From all the possible states in Γ_+, we only expect a subset in the physical spectrum of BPS bound states of the theory. These states are the following:

(1) Positive roots of G, α_+: they correspond to fractional branes [10, 27, 50], which are interpreted as D2-branes wrapping the 2-cycle α_+ in the homology lattice of the ALE space. For $\widehat{A}_{N-1}$ they have the form

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j\,, \quad i < j\,. \tag{4.4}$$

The central charge associated to a positive root is $Z_{\alpha_{ij}} = \zeta_i + \cdots + \zeta_j + i|j - i + 1|/N$, and we might have boundstates at threshold. We expect one bound state for each positive root, since if we go now to the orbifold with $B_{\alpha_+} = 0$, duality with heterotic string on $\mathbf{T}^4$ requires the existence of these states [51], and they cannot disappear from the spectrum. They correspond to the W-bosons of the enhanced gauge symmetry, and they have no degeneracy.

(2) Null roots of $\hat{G}$. They have the form $n\delta$, with $n > 0$. They correspond to a boundstate with n units of D0 charge. The mass is given by $m_{n\delta} = n/g_s$, so for $n > 1$, if they exist they are boundstates at threshold. Indeed, M/IIA duality requires their existence. A way to see this is to increase the coupling, so we are in M-theory on ALE $\times$ $\mathbf{S}^1$: we have to get all the KK modes, and we expect these states to be present also at $g_s = 0$.

(3) Roots of the form $\alpha_+ + n\delta$, with $n > 0$: they correspond to the KK modes of the W-bosons. They can be at threshold for special values of ζ. Again, M/IIA duality requires their existence.

(4) Roots of the form $-\alpha_+ + n\delta$, with $n > 0$. For $n = 1$, they are naturally interpreted as anti-D2 branes [52]. The states with $n > 1$ correspond to the KK modes of these anti-D2 branes.

Notice that the expected spectrum consists of the positive roots of the affine Lie algebra. (1), (3) and (4) correspond to the real roots:

$$\begin{aligned}\Delta_+^{\mathrm{real}}(\hat{G}) = &\Big\{\alpha_+ + n\delta,\ \alpha_+ \in \Delta_+(G),\ n \geq 0\Big\} \cup \\ &\cup \Big\{ -\alpha_+ + n\delta,\ \alpha_+ \in \Delta_+(G),\ n > 0\Big\}\,,\end{aligned} \tag{4.5}$$

while (2) corresponds to the imaginary roots:

$$\Delta_+^{\mathrm{im}}(\hat{G}) = \{n\delta |\, n > 0\}\,. \tag{4.6}$$

4.2 The quiver theory

The quiver diagram corresponding to D-branes on $\mathbb{C}^2/\mathbb{Z}_N$ is constructed in terms of the affine Dynkin diagram $\widehat{A}_r$, where $r = N - 1$ is the rank [16]. These are precisely the quivers that appear in the work of Kronheimer and Nakajima [53, 54]. The quiver diagram that one obtains is the following: take the corresponding affine Dynkin diagram, and denote the nodes by V_i, $i = 0, 1, \ldots, r$. These nodes are complex vector spaces of dimension n_i, so $V_i \simeq \mathbb{C}^{n_i}$, and the gauge group is $G = \prod_i U(n_i)$, with complexified gauge group $G_{\mathbb{C}} = \prod_i \mathrm{Gl}(n_i, \mathbb{C})$. For every node there are two arrows $B_{i,i+1}$, $B_{i,i-1}$ in opposite directions. The structure is the following:

$$\begin{tikzcd}
& & V_0 \arrow[dll, shift right, "B_{0,1}"'] \arrow[dr, shift left, "B_{0,r}"] & \\
V_1 \arrow[urr, shift right, "B_{1,0}"'] \arrow[r, shift left, "B_{1,2}"] & V_2 \arrow[l, shift left, "B_{2,1}"] \arrow[r, shift left, "B_{2,3}"] & \cdots \arrow[l, shift left, "B_{3,2}"] \arrow[r, shift left, "B_{r-1,r}"] & V_r \, . \arrow[l, shift left, "B_{r,r-1}"] \arrow[ul, shift left, "B_{r,0}"]
\end{tikzcd} \tag{4.7}$$

As we discussed above, there are complex and real equations. The complex equations come from the superpotential of the $\mathcal{N} = 2$ theory, which in the Higgs phase imposes the following relations:

$$B_{i+1,i} B_{i,i+1} = B_{i-1,i} B_{i,i-1} \, . \tag{4.8}$$

The real equations read:

$$D_k = B_{k-1,k} B^\dagger_{k-1,k} - B^\dagger_{k,k-1} B_{k,k-1} + B_{k+1,k} B^\dagger_{k+1,k} - B^\dagger_{k,k+1} B_{k,k+1} = \theta_k \, . \tag{4.9}$$

It is clear that life gets considerably simpler if we are left only with (4.8), so we will look for θ-stable $G_{\mathbb{C}}$-orbits of the solutions to the complex equations. Because of our previous remarks, each orbit will contain a solution to (4.9). The dimension vectors for the representations of this quiver are in one-to-one correspondence with the lattice Γ_+, after identifying the simple root α_i with the dimension vector whose entries are all zero except at the i-th place, where the entry is one: $n = (0, \ldots, 0, 1, 0, \ldots, 0)$ (we identify $0 \equiv N$). This is the dimension vector of the simple representation U_i, which has a single $\mathbb{C}$ at the i-th node, and all the arrows are set to zero.

Notice that, for this quiver, one has a supersymmetric vacuum only for $n = \delta = (1, 1, \ldots, 1)$, which describes the D0-brane. The other possible dimension vectors n correspond to fractional branes and combinations thereof. These fractional branes are interpreted as wrapped D2-branes (or anti D2-branes) with D0-charge, and are described by quasi-supersymmetric vacua: the D terms are not zero, but take a non zero, constant value which breaks supersymmetry. This gives the relation between the θ parameters and the FI terms stated in (2.6).

As a final remark, notice that the quiver (4.7), in contrast to the quivers considered in [3], is "non chiral", in the sense that there are two arrows with opposite orientations in each vertex. This is of course a consequence of $\mathcal{N} = 2$ supersymmetry. Non-chiral quivers can be regarded as a direct sum of two chiral quivers with opposite orientations, and they have been considered in some detail in [28].

4.3 The BPS spectrum from the quiver theory

Our purpose now is to find the θ-stable representations of the quiver (4.7), where θ is related to ζ by (2.6), and see that we recover all the physical expectations explained above.

We will denote the dimension vector of the representation by n. The first thing we want to know is the dimension of the moduli space of representations $\mathcal{M}_\theta(n)$. As shown in [28], this space contains as an open subset the θ-stable representations, and this open set has the expected complex dimension

$$2 - n^t \cdot C \cdot n \,, \tag{4.10}$$

where C is the generalized Cartan matrix of the quiver (in our case, the Cartan matrix of the affine $\widehat{\mathrm{SU}(N)}$). The first thing to do is to see which dimension vectors give a non-negative expected dimension. The relevant result here is due to Kac [55] and says that, for any Kac-Moody algebra with generalized Cartan matrix C, and for any vector of non-negative components n, $n^t \cdot C \cdot n = 2$ if and only if n is a real root of the algebra, and $n^t \cdot C \cdot n \leq 0$ if and only if n is an imaginary root. We then see that the only possible dimension vectors giving a non-negative dimension space of θ-stable representations are the positive roots Δ_+ of the Kac-Moody algebra. This is already in agreement with the physical results established above.

The next step is to analyze each of the positive roots and find the θ-stable representations, for different values of the Fayet-Iliopoulos parameters. This will prove the existence of bound states.

(1) *Positive roots of G.* For the simple roots α_i, the representation is given by U_i, $i = 0, \ldots, N-1$. Since they do not have any proper subrepresentation, they are θ-stable everywhere in moduli space, in a trivial sense.

For positive but not simple roots (the rest of the gauge multiplet) the story is far more interesting. Consider the simple case of a positive root given by the sum of two adjacent simple roots, $\alpha_i + \alpha_{i+1}$, $i = 0, 1, \ldots, N-1$, and we identify $0 \equiv N$. This corresponds to a D2-brane wrapping two adjacent $\mathbb{P}^1$'s. The complex equation (4.8) is easily solved in this case, and gives $B_{i,i+1} = 0$ or $B_{i+1,i} = 0$. Up to complex isomorphism, this gives two possible representations, with two different subobjects.

For $B_{i+1,i} = 0$ one has:

$$
\begin{tikzcd}
\cdots & 0 \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "\simeq"'] \arrow[r, shift right, "0"'] & 0 \arrow[l, shift right, "0"'] & \cdots \\
\cdots & 0 \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[u, "\simeq"'] \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & 0 \arrow[u, "0"'] \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & 0 \arrow[l, shift right, "0"'] & \cdots
\end{tikzcd}
\tag{4.11}
$$

while for $B_{i,i+1} = 0$ one has:

$$
\begin{tikzcd}
\cdots & 0 \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "\simeq"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & 0 \arrow[l, shift right, "0"'] & \cdots \\
\cdots & 0 \arrow[r, shift right, "0"'] & 0 \arrow[u, "0"'] \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[u, "\simeq"'] \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & 0 \arrow[l, shift right, "0"'] & \cdots
\end{tikzcd}
\tag{4.12}
$$

The meaning of these diagrams is as follows: the top row is the original representation, with 0 and $\simeq$ denoting the zero and the identity map, respectively. Physically, these two possibilities mean that the chiral fields $B_{i,i+1}$, $B_{i+1,i}$ have a zero or non-zero vev. When the vev is non zero, we perform a complex gauge transformation to set it equal to the identity map. The bottom row is the subrepresentation. All the maps from the bottom to the top row have to be injective, by definition of subrepresentation. Checking that these diagrams commute is extremely easy. The representation in (4.11) is θ-stable if $\theta_i > 0$, and the second representation (4.12) is stable if $\theta_i < 0$. If $\theta_i = 0$, both are semistable, and both are S-equivalent to the direct sum $U_i \oplus U_{i+1}$, which is the graded representation. In our general discussion we stressed that stability allows us to bypass solving the D-flatness equations; in this case it is immediate to check the equivalence of the two approaches: θ-stable representations are in one-to-one correspondence with the solutions to the real equations, which in this case are simply:

$$|B_{i+1,i}|^2 - |B_{i,i+1}|^2 = \theta_i \,. \tag{4.13}$$

For $\theta_i > 0$, we have $|B_{i+1,i}| = \sqrt{\theta_i}$ and $B_{i,i+1} = 0$, and for $\theta_i < 0$, we have $B_{i+1,i} = 0$ and $|B_{i,i+1}| = \sqrt{-\theta_i}$, in agreement with the stability analysis. For $\theta_i = 0$, the solution is $B_{i+1,i} = B_{i,i+1} = 0$, i.e. the graded representation.

Physically, the semistable point $\theta_i = 0$ corresponds to a line of marginal stability when $\zeta_i = \zeta_{i+1}$ in the space of FI terms, as one can see using (2.6). One can easily check, using (4.1), that on this line

$$\|Z_{\alpha_i+\alpha_{i+1}}\| = \|Z_{\alpha_i}\| + \|Z_{\alpha_{i+1}}\| \,, \tag{4.14}$$

in agreement with the structure of the graded representation $U_i \oplus U_{i+1}$. However, there is no decay of the state through this line. This is because at both sides of the line there is a stable representation with dimension $n = \alpha_i + \alpha_{i+1}$, as we have just seen. Therefore, in these models *there are lines of marginal stability which do not give a decay.* As expected, this is due to the fact that our quiver is non chiral, which is in turn a consequence of having sixteen supercharges in the bulk. Notice that the "binding" of the BPS states, in the quiver picture, corresponds very precisely to the maps between the nodes.

For the more general positive roots $\alpha_i + \cdots + \alpha_j$ the story is very similar. There are in general $j - i$ lines of marginal stability in the ζ-space, and there are no decays through them: the BPS state exists on both sides of the lines.

(2) *Null roots of* $\hat{G}$. The moduli space $\mathcal{M}_\zeta(\delta)$ has real dimension 4, and it is the ALE space itself, as it follows from the results of [53, 54]. Notice that there are lines of marginal stability in the space of FI terms. For example, for $\widehat{\mathrm{SU}(3)}$, the lines will be at $\zeta_1 = 0$, $\zeta_2 = 0$, and $\zeta_1 + \zeta_2 = 0$, as one can easily check by using the mass formula. At these lines, the representations will be only semistable. The moduli space $\mathcal{M}_\zeta(\delta)$ can be understood as a resolution of singularities in the following sense [28]. For $n = \delta$, all the maps in (4.7) are just complex numbers. If one defines:

$$
\begin{gathered}
x = B_{1,0} B_{2,1} \cdots B_{n,n-1} B_{0,n} \,, \\
y = B_{n,0} B_{n-1,n} \cdots B_{1,2} B_{0,1} \,, \qquad z = B_{1,0} B_{0,1} \,,
\end{gathered} \tag{4.15}
$$

then the complex equations (4.8) imply that $xy = z^{n+1}$. This is the equation that defines the quotient $\mathbb{C}^2/\mathbb{Z}_N$ as a subspace of $\mathbb{C}^3$. Therefore, one has a map:

$$
\pi : \mathcal{M}_\zeta(\delta) \longrightarrow \mathbb{C}^2/\mathbb{Z}_N \,. \tag{4.16}
$$

It can be seen that this map is an isomorphism outside the inverse image of the origin $\pi^{-1}(0)$. This set is called the *exceptional set*, and it is given by $N-1$ $\mathbb{P}^1$'s (the exceptional divisors). We will denote them by Σ_i, $i = 1, \ldots, N-1$.

The BPS spectrum is given by the L^2-cohomology of the ALE space $H^*_{L^2}(\mathcal{M}_\zeta(\delta))$, which is concentrated in the middle dimension [56]. A convenient choice of basis is given by the Poincaré duals to the Σ_i, $\omega_i = [\Sigma_i]$. These cohomology classes satisfy [54, 28]:

$$
\int_{\Sigma_k} \omega_j = C_{jk} \,, \tag{4.17}
$$

where C_{jk} is the Cartan matrix of the Lie algebra. We then find that there are $N-1$ BPS states with charge δ.

When the dimension vector is $n\delta$, the moduli space is the n-th symmetric product of the ALE, and at a generic point, the automorphism group is $(\mathbb{C}^*)^n$, so the representation is not Schur, and therefore it cannot be stable. We have to restrict to the small diagonal [6], which is a copy of the ALE space itself. The representations in the small diagonal are only semistable, but one expects that the cohomology (which is given again by the two-forms ω_i, $i = 1, \dots, N-1$) corresponds to bound states at threshold. Summarizing, for any imaginary root $n\delta$ with $n > 0$, the BPS spectrum is given by $N-1$ states which are in one-to-one correspondence with the L^2-cohomology of the ALE space.

(3) *The rest of the roots.* The rest of the roots have the form $\pm\alpha_+ + n\delta$, $n > 0$. The moduli space has dimension zero, so $\mathcal{M}_\theta(\pm\alpha_+ + n\delta)$ is a set of points, each of them giving one BPS state (the space could be empty). In the next subsection, we will analyze in detail a particular case in affine SU(3) and see that it gives the right degeneracy: one θ-stable representation for each value of θ, giving a single BPS state. It also shows a rich structure of lines of marginal stability and will illustrate our discussion of graded representations.

4.4 A detailed case study

We will now study in detail the representations with dimension $n = \alpha_1 + \delta$ for $\widehat{\mathrm{SU}(3)}$. This will illustrate very well the richness as well as the computability of the procedure. The quiver representation has the diagram:

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dl, shift right, "b"'] \arrow[dr, shift left, "c"] & \\
\mathbb{C}^2 \arrow[ur, shift right, "\tilde{b}"'] \arrow[rr, shift left, "a"] & & \mathbb{C}, \arrow[ul, shift left, "\tilde{c}"] \arrow[ll, shift left, "\tilde{a}"]
\end{tikzcd} \tag{4.18}$$

where a, $\tilde{a}$, b and $\tilde{b}$ are regarded as two-component vectors, and one has for example $a \cdot (z_1, z_2) = a_1 z_1 + a_2 z_2$, $\tilde{a} \cdot z = (\tilde{a}_1 z, \tilde{a}_2 z)$, and so on. The complex equations (4.8) can be written as follows:

$$\begin{gathered} a \cdot \tilde{a} = b \cdot \tilde{b} = c\tilde{c}\,, \\ b_1\tilde{b}_1 = a_1\tilde{a}_1\,, \qquad b_2\tilde{b}_2 = a_2\tilde{a}_2\,, \qquad b_1\tilde{b}_2 = \tilde{a}_1 a_2\,, \qquad b_2\tilde{b}_1 = \tilde{a}_2 a_1\,. \end{gathered} \tag{4.19}$$

and we want to find the θ-stable representations, for arbitrary values of θ. Since θ-stable representations are Schur, a good starting point is to look for Schur representations.

From the definition, we have to consider all the possible endomorphism of the generic representation (4.18). These are given by $\lambda_1, \lambda_2 \in \mathbb{C}^*$, acting on the $\mathbb{C}$ nodes, and a matrix $S \in \mathrm{Gl}(2, \mathbb{C})$ acting on $\mathbb{C}^2$. Commutativity of all the diagrams gives the equations:

$$\begin{gathered} (\lambda_1 - \lambda_2)c = (\lambda_1 - \lambda_2)\tilde{c} = 0\,, \\ S^t \cdot a = \lambda_1 a\,, \qquad S \cdot \tilde{a} = \lambda_1 \tilde{a}\,, \qquad S \cdot b = \lambda_2 b\,, \qquad S^t \cdot \tilde{b} = \lambda_2 \tilde{b}\,. \end{gathered} \tag{4.20}$$

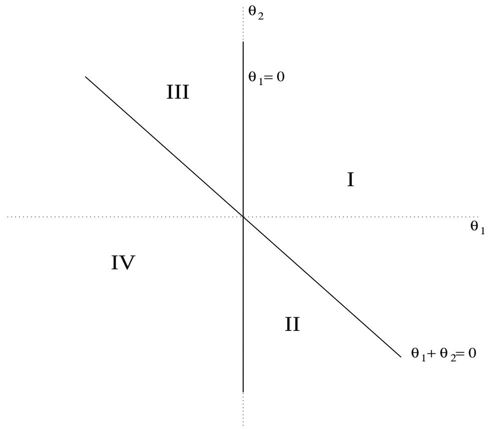


Figure 3: Lines of marginal stability for $\alpha_1 + \delta$.

Schur representations are such that they force $\lambda_1 = \lambda_2 = \lambda$ and $S = \lambda \cdot \mathbf{1}_{2\times 2}$. A careful analysis of these equations shows that, for every value of θ, there is only one Schur representation (up to complex isomorphism) which is θ-stable and solves the complex equations. These solutions are as follows:

(a) For $\theta_1 > 0$, $\theta_1 + \theta_2 > 0$ (region I in figure 3), the stable representation is:

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, shift right, "i_2"'] \arrow[dr, shift left, "\simeq"'] & \\ \mathbb{C}^2 \arrow[ur, shift right, "0"'] \arrow[rr, shift left, "0"] & & \mathbb{C}, \arrow[ul, shift right, "0"'] \arrow[ll, shift left, "i_1"] \end{tikzcd} \tag{4.21}$$

where i_p denotes the inclusion in the p-th factor. The subrepresentations have dimensions $(2,0,0)$, $(2,1,0)$ and $(1,1,0)$ (in the notation $n = (n_1, n_2, n_0)$). When $\theta_1 > 0$ but $\theta_1 + \theta_2 < 0$ (region II), the θ-stable representation is like (4.21), but with c and $\tilde{c}$ exchanged. The subrepresentations have dimensions $(2,0,0)$, $(1,0,1)$ and $(2,0,1)$.

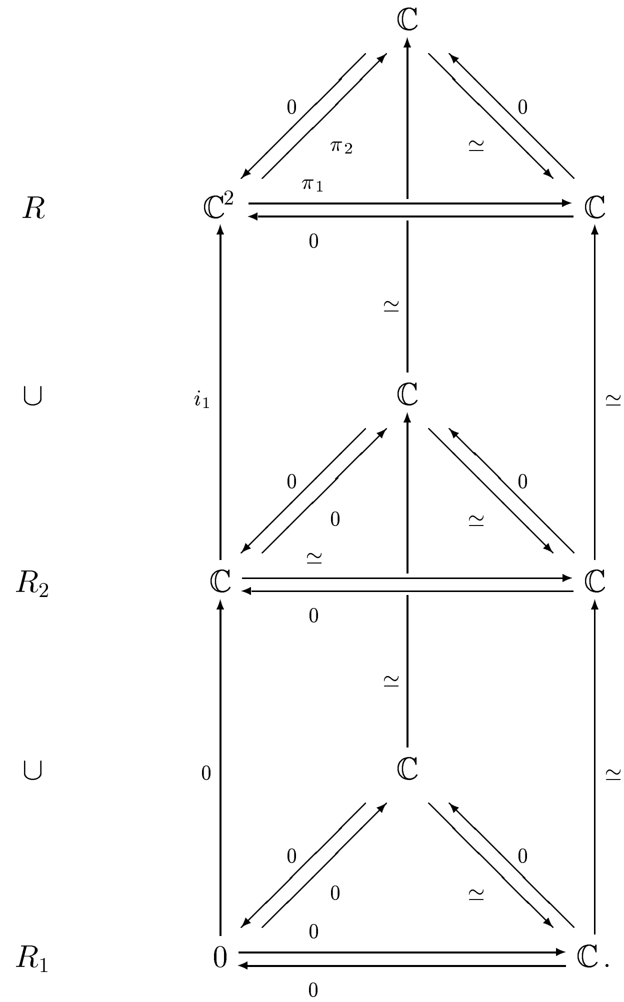
(b) For $\theta_1 < 0$, and $\theta_1 + \theta_2 > 0$ (region III), the θ-stable representation is

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "\simeq"'] & \\ \mathbb{C}^2 \arrow[ur, shift right, "\pi_2"'] \arrow[rr, shift left, "\pi_1"] & & \mathbb{C}, \arrow[ul, shift right, "0"'] \arrow[ll, shift left, "0"] \end{tikzcd} \tag{4.22}$$

where π_p denote the projection onto the p-th factor. The subrepresentations of (4.22) have dimensions $(0,1,0)$, $(0,1,1)$, $(1,1,0)$ and $(1,1,1)$. Finally, for

$\theta_1 < 0$ and $\theta_1 + \theta_2 < 0$ (region IV in figure 3), the θ-stable representation is like (4.22), but again with c and $\tilde{c}$ exchanged. The subrepresentations have dimensions $(0,0,1)$, $(0,1,1)$, $(2,0,1)$ and $(1,1,1)$.

We see that there are two lines of marginal stability, at $\theta_1 = 0$ and at $\theta_1 + \theta_2 = 0$, but again no decay takes place through them, since θ-stable representations exist on both sides of the lines. Let's analyze in more detail what happens on these lines. The representations that we have described are only semistable there, and they will have a non-trivial Jordan-Hölder filtration. Since we have listed all the subobjects, we can construct the filtrations very easily, for the different lines of marginal stability. Consider first the half-line $\theta_1 = 0$, $\theta_2 > 0$, which separates region I from region III. The representation (4.22) becomes semistable and has the following filtration:



(4.23)

Notice that all the representations in the filtration satisfy $\sum_v n_v \theta_v = 0$ on the line $\theta_1 = 0$. The last representation is stable, since its only subobject has dimension $(0,1,0)$. The two quotients involved in the filtration are isomorphic to the simple representation U_1, therefore they are stable. This shows that the above is an admissible Jordan-Hölder filtration. The graded representation is:

$$
\begin{tikzcd}
& \mathbb{C} \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "\simeq"] & \\
0 \arrow[ur, shift right, "0"'] \arrow[rr, shift left, "0"] & & \mathbb{C} \arrow[ll, shift left, "0"] \arrow[ul, shift left, "0"]
\end{tikzcd}
\oplus\ 2
\begin{tikzcd}
& 0 \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "0"] & \\
\mathbb{C} \arrow[ur, shift right, "0"'] \arrow[rr, shift left, "0"] & & 0\,. \arrow[ll, shift left, "0"] \arrow[ul, shift left, "0"]
\end{tikzcd}
\tag{4.24}
$$

The θ-stable representation on the other side of the line (region I) is (4.21). It also becomes semistable on the line, and its graded representation is also (4.24). Therefore, the θ-stable representations on both sides of the line of marginal stability become S-equivalent on it.

As a further example, let's consider the line $\theta_1 + \theta_2 = 0$, and $\theta_1 < 0$, which separates region I from region II. The representation (4.22) becomes again semistable, but in this case it has a very different Jordan-Hölder filtration:

$$
\begin{tikzcd}
& & \mathbb{C} \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "\simeq"] & \\
R & \mathbb{C}^2 \arrow[ur, shift right, "\pi_2"'] \arrow[rr, shift left, "\pi_1"] & & \mathbb{C} \arrow[ll, shift left, "0"] \arrow[ul, shift left, "0"] \\
\cup & & 0 \arrow[uu, "0"] \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "0"] & \\
R_1 & \mathbb{C} \arrow[uu, "i_1"] \arrow[ur, shift right, "0"'] \arrow[rr, shift left, "\simeq"] & & \mathbb{C}\,. \arrow[uu, "\simeq"'] \arrow[ll, shift left, "0"] \arrow[ul, shift left, "0"]
\end{tikzcd}
\tag{4.25}
$$

The graded representation is in this case:

$$
\begin{tikzcd}
& 0 \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "0"] & \\
\mathbb{C} \arrow[ur, shift right, "0"'] \arrow[rr, shift left, "\simeq"] & & \mathbb{C} \arrow[ll, shift left, "0"] \arrow[ul, shift left, "0"]
\end{tikzcd}
\oplus
\begin{tikzcd}
& \mathbb{C} \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "0"] & \\
\mathbb{C} \arrow[ur, shift right, "\simeq"'] \arrow[rr, shift left, "0"] & & 0\,. \arrow[ll, shift left, "0"] \arrow[ul, shift left, "0"]
\end{tikzcd}
\tag{4.26}
$$

Again, the θ-stable representation on the other side of the line has the same graded representation on it, and it becomes S-equivalent to (4.22).

The above graded representations correspond nicely to our physical expectations based on the central charges. Using (2.6), one finds that θ_1, θ_2 are related to the FI parameters as follows:

$$\theta_1 = \frac{3}{4}\zeta_1 \,, \qquad \theta_2 = \zeta_2 - \frac{1}{4}\zeta_1 \,. \tag{4.27}$$

Therefore the lines of marginal stability are $\zeta_1 = 0$, $\zeta_1 + 2\zeta_2 = 0$. When $\zeta_1 = 0$, the central charge satisfies:

$$\|Z_{\alpha_1+\delta}\| = 2\|Z_{\alpha_1}\| + \|Z_{-\alpha_1+\delta}\| \,, \tag{4.28}$$

and the decomposition is precisely the one in (4.24). Also, for $\zeta_1 + 2\zeta_2 = 0$ one finds

$$\|Z_{\alpha_1+\delta}\| = \|Z_{\alpha_1+\alpha_2}\| + \|Z_{-\alpha_2+\delta}\| \,, \tag{4.29}$$

as dictated by (4.26). This shows very clearly that the decompositions given by the graded representations are the minimal ones, since the subobjects appearing in the direct sum are strictly stable on the line.

4.5 Computation of the BPS algebra

The spectrum that we have found matches perfectly with the generators of the positive part of the Kac-Moody algebra: the r states with charges $n\delta$ (which have the charges of n-D0 branes) give the generators H^i_n, $i = 1\cdots r$, $n > 0$. The state of a D2 wrapping α_+ with n D0-branes, which has charge $\alpha_+ + n\delta$, corresponds to $E^{\alpha_+}_n$, with $n \geq 0$. The anti-D2 branes together with their KK modes give the generators $E^{-\alpha_+}_n$, $n > 0$. These generators give a subalgebra of the full Kac-Moody algebra $\widehat{A}_r$, with the following relations (we follow the conventions of [57]):

$$\begin{aligned}
\left[H^i_m, H^j_n\right] &= 0 \,, \quad m,n > 0 \,, \\
\left[H^i_m, E^\alpha_n\right] &= \alpha^i E^\alpha_{m+n} \,, \quad m > 0 \,,\ n \geq 0 \,,\ \alpha \in \Delta_+ \,, \\
\left[E^\alpha_m, E^{-\alpha}_n\right] &= -\alpha \cdot H_{m+n} \,, \quad m \geq 0 \,,\ n > 0 \,,\ \alpha \in \Delta_+ \,, \\
\left[E^\alpha_m, E^\beta_n\right] &= \epsilon(\alpha,\beta) E^{\alpha+\beta}_{m+n} \,, \quad \alpha, \beta, \alpha+\beta \in \Delta_+ \,,
\end{aligned} \tag{4.30}$$

where, in the last commutator, m, n are non negative or positive when α, β are positive or negative, respectively. The $\epsilon(\alpha, \beta)$ are the $\mathbb{Z}_2$ cocycles and take values ± 1. Notice that there are many consistent choices of $\epsilon(\alpha, \beta)$, which turn out to be related to choices of orientation of the quiver diagram [57, 39]. Finally, the roots α in the l.h.s. of (4.30) are understood as vectors in the Dynkin basis with components α^i, and one has for example for the simple roots $(\alpha_k)^i = C_{ki}$, so that $\alpha_k \cdot H_n = C_{ki} H^i_n$.

Due to heterotic/type-IIA duality, the algebra of D2-D0 BPS states that we have considered should be the subalgebra (4.30) of the full Kac-Moody algebra. This has been explicitly checked in [6] by a vertex operator computation in the heterotic side. We will now compute the BPS algebra in a few cases starting from the definition in terms of representations of quivers.

Consider first the positive roots of the simple Lie algebra A_r, and the simple case of $n_1 = \alpha_i$, $n_2 = \alpha_{i+1}$ and $n_3 = \alpha_i + \alpha_{i+1}$. The moduli spaces of representations $\mathcal{M}_\zeta(n_i)$ are nonempty, consist of a single point, and this point is a stable representation away from the marginal stability line $\zeta_i = \zeta_{i+1}$, as we saw in section 4. The correspondence variety is also a point. When $\zeta_i > \zeta_{i+1}$, this point corresponds to the short exact sequence:

$$
\begin{tikzcd}
R_2 & \cdots & 0 \arrow[r, shift right, "0"'] & 0 \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] & 0 \arrow[l, shift right, "0"'] & \cdots \\
R_3 \arrow[u] & \cdots & 0 \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] \arrow[u, "0"'] & \mathbb{C} \arrow[l, shift right, "\simeq"'] \arrow[r, shift right, "0"'] \arrow[u, "\simeq"'] & 0 \arrow[l, shift right, "0"'] & \cdots \\
R_1 \arrow[u] & \cdots & 0 \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] \arrow[u, "\simeq"'] & 0 \arrow[l, shift right, "0"'] \arrow[r, shift right, "0"'] \arrow[u, "0"'] & 0 \arrow[l, shift right, "0"'] & \cdots
\end{tikzcd}
\qquad (4.31)
$$

while for $\zeta_i < \zeta_{i+1}$, one has the sequence with the roles of U_i and U_{i+1} inverted. Recall that, when the correspondence variety is a point, one has simply to count short exact sequences. We then find:

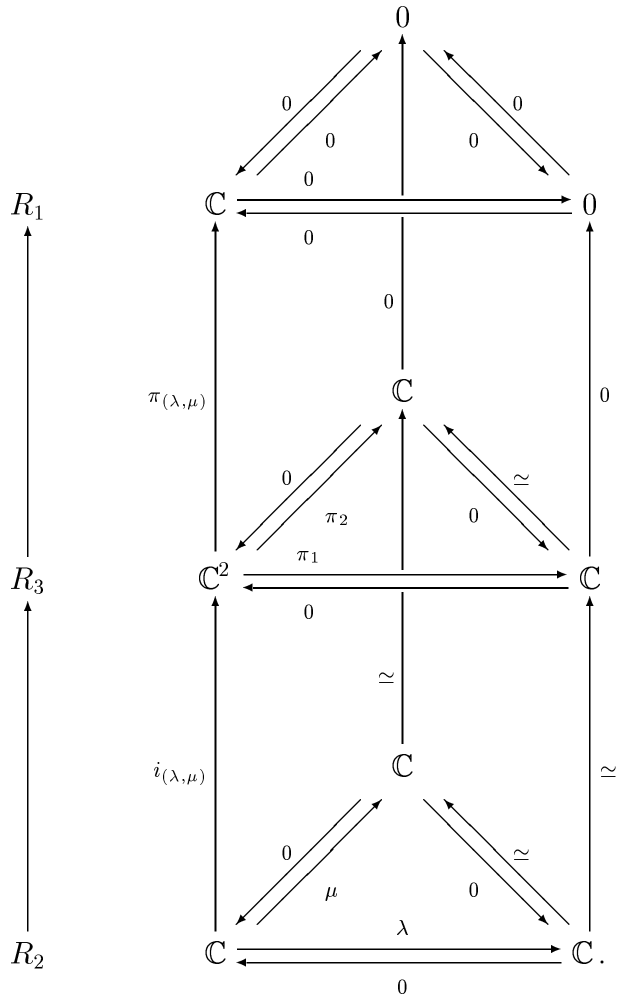
$$[E_0^{\alpha_i}, E_0^{\alpha_{i+1}}] = \mathrm{sgn}(\zeta_i - \zeta_{i+1}) E_0^{\alpha_i+\alpha_{i+1}} \,. \tag{4.32}$$

Notice that the sign $\epsilon(\alpha_i, \alpha_{i+1}) = \mathrm{sgn}(\zeta_i - \zeta_{i+1})$ depends on the region of the moduli space we are considering. This suggests that the different regions of stability in the ζ-space are related to choices of orientation of the underlying diagram, and in fact one can check that this is the case in concrete examples.

The computation of the structure constants for the other positive roots of A_r follows the same lines, and it is rather easy. In all cases one finds the last relation of (4.30), and the sign depends again on the precise values on the region of the moduli space of FI parameters that we are sitting in. We would like to make two remarks: first, this computation is essentially the computation of the Ringel-Hall algebra associated to the A_r Dynkin diagram (see for example the recent paper [40]). Second, it is remarkable that such a simple computation already captures an important part of the physics associated to the BPS states in this compactification, namely the emergence of Lie algebra structures.

Let's now come to the part of the algebra involving positive roots of the full Kac-Moody algebra. As in the previous subsection, we will make a sample computation in the case of $\widehat{\mathrm{SU}(3)}$. This computation will involve non-trivial correspondence varieties of positive dimension, and prove the power of the quiver approach in performing actual computations.

Consider first the case $n_1 = \alpha_1$, $n_2 = \delta$, $n_3 = \alpha_1 + \delta$. We will choose a ζ such that $\zeta_1 < 0$, $\zeta_2 < 0$, and $\zeta_1 + 2\zeta_2 < 0$ (this choice is made for convenience, since in that case the arguments in [28] make possible to identify very quickly the correspondence variety). In this region, the stable representative of $\alpha_1 + \delta$ is given by (4.22), but with the arrows on the right exchanged (i.e. in the notation of subsection 4.4, c is exchanged with $\tilde{c}$). In order to construct the short exact sequence, we need n_1 or n_2 to be dimension vectors of subrepresentations. In this case, there are no subrepresentations with dimension $(1,0,0)$, therefore $\mathcal{C}(n_1, n_2; n_3) = 0$. It turns out that there is a whole family of embeddings of representations with dimension vector $(1,1,1)$ into the θ-stable representation with dimension $\alpha_1 + \delta$. The short exact sequence that one obtains is:



(4.33)

In this diagram, μ and λ are complex numbers, and the maps $i_{(\lambda,\mu)}$ and $\pi_{(\lambda,\mu)}$ are given by

$$i_{(\lambda,\mu)}(z) = (\lambda\, z, \mu\, z)\,, \qquad \pi_{(\lambda,\mu)}(z_1, z_2) = \mu z_1 - \lambda z_2\,, \tag{4.34}$$

in such a way that $\pi_{(\lambda,\mu)} i_{(\lambda,\mu)} = 0$, as required by exactness. The space of short exact sequences is then parametrized by the pairs (λ, μ). But we have to take into account two things: first of all, $(\lambda, \mu) \neq (0,0)$, since in that case the first representation in the short exact sequences would be decomposable, contradicting stability. On the other hand, one has to quotient by the complexified gauge group acting on the representation. We then consider endomorphisms $\alpha_i : \mathbb{C} \to \mathbb{C}$, where $\alpha_i \in \mathbb{C}^*$. $i = 1, 2, 0$, and they act by multiplication on the nodes of the representation. The space of complexified gauge transformations actually has dimension two, so we have to get rid of one extra degree of freedom. Since we want to study the effect of gauge transformations on the maps (λ, μ), we fix this extra degree by requiring that the transformed representations have the same form as in (4.33), i.e. that they leave the arrows on the right unchanged in the representations with dimension $(1,1,1)$. This imposes $\alpha_0 = \alpha_2$. Therefore, the action of the complexified gauge group is:

$$(\lambda, \mu) \longrightarrow (\alpha\lambda, \alpha\mu)\,, \tag{4.35}$$

where $\alpha = \alpha_1^{-1}\alpha_2 \in \mathbb{C}^*$. The space that parameterizes the short exact sequences is nothing but the projective space $\mathbb{P}^1$. We then find:

$$\mathcal{C}(n_2, n_1; n_3) = \mathbb{P}^1\,. \tag{4.36}$$

Another way to see this is to notice that the space of short exact sequences (4.33) is in one-to-one correspondence with the space of embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ up to complex isomorphisms (the maps that we have denoted by $i_{(\lambda,\mu)}$). This space is nothing but the grassmannian $G(1,2) \simeq \mathbb{P}^1$. Notice that, from this point of view, $(\lambda, \mu) = (0,0)$ is forbidden by injectivity of $i_{(\lambda,\mu)}$.

In order to compute the product, we have to be more explicit about this space. First of all, notice that this $\mathbb{P}^1$ can be identified in a natural way with a subspace of the space of θ-stable representations with dimension $(1,1,1)$, which is the ALE space. Moreover, the $\mathbb{P}^1$ is contained in the exceptional set, as one can see using (4.15). We can go further and identify this $\mathbb{P}^1$ with one of the Σ_i's. To do this, we use [28, theorem 5.10], which gives a very concrete recipe to identify representations in $\pi^{-1}(0)$ with the exceptional $\mathbb{P}^1$'s. The recipe is the following: consider the representation R in question, and construct a filtration:

$$0 \subset R_1 \subset \cdots \subset R_p \subset R\,, \tag{4.37}$$

in such a way that all the quotients R_i/R_{i-1} are isomorphic to simple representations U_i. The existence of such a decomposition is guaranteed by the fact that $R \in \pi^{-1}(0)$. Moreover, $R \in \Sigma_j$ if and only if $R/R_p \simeq U_j$. Notice that the above is not a Jordan-Hölder filtration, since it does not take into account θ-stability.[12] It corresponds in

[12]However, it can be regarded as the Jordan-Hölder filtration of R when $\theta = 0$.

fact to the decomposition of R into simple pieces. In our example, the construction of the filtration is rather easy. It has $p = 2$, $R_1 = U_0$ and R_2 is

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, shift right, "0"'] \arrow[dr, shift left, "0"] & \\
0 \arrow[ur, shift right, "0"'] \arrow[rr, shift left, "0"] & & \mathbb{C} \, . \arrow[ul, shift left, "\simeq"] \arrow[ll, shift left, "0"]
\end{tikzcd} \tag{4.38}$$

Therefore, $R/R_2 = U_1$ and the correspondence variety (4.36) is in fact Σ_1. Finally, using (4.17), one finds

$$\left[H_1^i, E_0^{\alpha_1}\right] = C_{i1} E_1^{\alpha_1} \,, \tag{4.39}$$

and since $\alpha_1^i = C_{i1}$, we find perfect agreement with (4.30). Using the above information, one can also compute the BPS product for $n_1 = \alpha_1$, $n_2 = -\alpha_1 + \delta$, $n_3 = \delta$, and obtain

$$\left[E_0^{\alpha_1}, E_1^{-\alpha_1}\right] = -\sum_i C_{1i} H_1^i \,. \tag{4.40}$$

This is again in agreement with (4.30).

In conclusion, we see that representations of quivers give a framework for computing BPS algebras in an efficient way, since the use of linear algebra makes possible to give precise descriptions of the correspondence variety of [6].

5. Conclusions

In this paper we have used the notions of stability and S-equivalence to clarify the structure of D-brane moduli spaces in different regions of the Kähler moduli space. In particular, we have seen that one can give a detailed recipe to obtain decay products on lines of marginal stability by using graded objects. We have also shown that the algebra of BPS states and the correspondence conjecture of [6] can be extended to the orbifold point, and formulated in terms of representations of quivers. This gives a very useful framework to study and compute BPS algebras. All these ideas have been tested in the rather simple example of type-IIA theory on an ALE space, where we can compare the results obtained from our constructions with expectations based on the use of various dualities.

An obvious direction for future work is to study lines of marginal stability and algebras of BPS states for Calabi-Yau orbifolds. The immediate candidate is $\mathbb{C}^3/\mathbb{Z}_3$, since its BPS spectrum near the orbifold has been studied in [3]. The adjacency matrix of the corresponding quiver gives a generalized Cartan matrix of the hyperbolic type, and it is likely that the BPS algebra of fractional branes will turn out to be related to the hyperbolic Kac-Moody algebra associated to the Cartan matrix. Also, as mentioned in [3] and fully worked out in [58], quiver techniques can be applied to Gepner points, giving a handle on compact Calabi-Yau manifolds. We also expect the algebra of BPS states to be related to the Kac-Moody algebra defined by the corresponding Cartan matrix. This would be give a very interesting generalization of the Nakajima construction [41] providing a geometrical setting for non-affine Kac-Moody algebras.

A different approach to some of these questions has been proposed recently in [59, 60, 61], where Landau-Ginzburg (LG) duals are constructed for many compact and non-compact Calabi-Yau manifolds, and explicit descriptions for the D-branes on both sides are provided. It would be very interesting to see what are the mirror descriptions of BPS spectra, lines of marginal stability, decay products, and BPS algebras, using the LG approach. This may be helpful as well to obtain the right category of objects at arbitrary points of moduli space.

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A. Π-stability and abelian categories

In this appendix we prove that the subcategory of Π-semistable D-branes with a fixed grading φ at a given point in moduli space u, is abelian. We use arguments valid for small difference of gradings.

First we give a couple of informal definitions. A category is just a set of objects and maps between them. A category is additive when there is some notion of sum in it, such that the sum of two objects is also an object in the same category. Finally, a category is abelian if it is additive and has the notion of kernel and cokernel for the maps in the category.

Let us call $\mathcal{A}$ the category of all branes satisfying the holomorphic constraint. As we explained in the text, we assume $\mathcal{A}$ is abelian. Consider the subcategory $A_\varphi(u)$ of D-branes with fixed grading φ, that are Π-semistable at the point in moduli space u. Recall that the D-brane E is Π-semistable if $\varphi(E) \geq \varphi(E')$ for all the subobjects E' of E at u.

First let's prove that $A_\varphi(u)$ is additive. For that we need to define the notion of sum, and check that it takes two elements of $A_\varphi(u)$ to another element in $A_\varphi(u)$. The notion of sum is given by exact sequences. If we have three D-branes E, F and G forming the exact sequence

$$0 \longrightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \longrightarrow 0\,, \tag{A.1}$$

and E and G are in $A_\varphi(u)$, we want to prove that F is also in $A_\varphi(u)$, i.e. that $\varphi(F) \geq \varphi(F')$ for all the subobjects F' of F at u. Take such an F'. Let $G' = \psi(F')$, and define E' by the short exact sequence

$$0 \longrightarrow E' \longrightarrow F' \longrightarrow G' \longrightarrow 0\,, \tag{A.2}$$

in other words, $\phi(E') = \mathrm{Ker}\,\phi|_{F'}$. E' and G' are subobjects of E and G, respectively, and by assumption $\varphi(E') \leq \varphi(E)$ and $\varphi(G') \leq \varphi(G)$. Here is where we restrict to small difference of gradings. This just means that the central charges are just complex numbers, but in this case the phases satisfy a convexity condition

$$\varphi(F') = x\varphi(E') + (1-x)\varphi(G')\,, \tag{A.3}$$

for some $0 \leq x \leq 1$. Using that $\varphi(E') \leq \varphi(E)$ and $\varphi(G') \leq \varphi(G)$ we obtain $\varphi(F') \leq \varphi(F)$.

Now we want to prove that $A_\varphi(u)$ is abelian. This requires that every map between objects has a kernel and a cokernel in the category. Consider the map $f : E \to F$, with E and F in $A_\varphi(u)$. First we argue that $\varphi(\mathrm{Im}\,f) = \varphi$. On one hand, $\mathrm{Im}\,f$ is a subobject of F, so $\varphi(\mathrm{Im}\,f) \leq \varphi$ by assumption. On the other hand, we have the exact sequence $0 \to \mathrm{Ker}\,\varphi \to E \to \mathrm{Im}\,\varphi \to 0$, so $\varphi(\mathrm{Im}\,f) \geq \varphi$, and our claim follows. From the exact sequence we just wrote it follows that $\varphi(\mathrm{Ker}\,f) = \varphi$. Furthermore, $\mathrm{Ker}\,f$ has to be Π-semistable, because every subobject of $\mathrm{Ker}\,f$ is also subobject of E, so a subobject destabilizing $\mathrm{Ker}\,f$ would also destabilize E. A dual of this argument shows that Coker f is also Π-semistable.

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